

Stopping time property of thresholds of Storey-type FDR procedures

Xiongzhi Chen^{*} and Rebecca W. Doerge[†]

Abstract

For multiple testing, we introduce Storey-type FDR procedures and the concept of “regular estimator of the proportion of true nulls”. We show that the rejection threshold of a Storey-type FDR procedure is a stopping time with respect to the backward filtration generated by the p-values and that a Storey-type FDR estimator at this rejection threshold equals the pre-specified FDR level, when the estimator of the proportion of true nulls is regular. These results hold regardless of the dependence among or the types of distributions of the p-values. They directly imply that a Storey-type FDR procedure is conservative when the null p-values are independent and uniformly distributed.

Keywords: False discovery rate, rejection threshold, stopping time property, Storey-type FDR procedure.

MSC 2010 subject classifications: Primary 62C25, 62M99.

1 Introduction

To control false discovery rate (FDR, [Benjamini and Hochberg, 1995](#)) in multiple testing, it is crucial to show the conservativeness of an FDR procedure, i.e., that the FDR of an FDR procedure is no larger than a pre-specified level. To show the conservativeness of their FDR procedures, [Storey et al. \(2004\)](#) used the stopping time property (STP) of the rejection threshold that was claimed in their Lemma 4, and [Liang and Nettleton \(2012\)](#) quoted this lemma as their

^{*}Corresponding author. Center for Statistics and Machine Learning, and Lewis-Sigler Institute for Integrative Genomics, Princeton University, Princeton, NJ 08544. Email: xiongzhi@princeton.edu.

[†]Departments of Statistics and Agronomy, Purdue University, West Lafayette, IN 49707. Email: doerge@purdue.edu.

Lemma 5 and used it to prove their Theorem 7. They assumed that the p-values corresponding to the true null hypotheses (i.e., null p-values) are independent and uniformly distributed. However, none of them provided a formal proof of the STP of the rejection thresholds, and without the STP the optional stopping theorem (see, e.g., [Karatzas and Shreve, 1991](#)) can not be applied and the martingale based approach in [Storey et al. \(2004\)](#) and [Liang and Nettleton \(2012\)](#) fails.

In this article, we introduce Storey-type FDR procedure in [Section 2](#) as an extension of Storey's procedure in [Storey et al. \(2004\)](#), introduce the concept of regular estimator of the proportion of true nulls, and show in [Section 3](#) that the STP of the rejection threshold of a Storey-type FDR procedure is generic when the estimator of the proportion of true nulls is regular. This provides a formal justification of the use of STP of the rejection thresholds in [Storey et al. \(2004\)](#) and, if needed, in [Chen and Doerge \(2014\)](#). However, it also implies that the rejection threshold of an adaptive FDR procedure may not be a stopping time when the estimator of proportion true nulls is not regular. Further, we show that a Storey-type FDR estimator at the threshold of its corresponding Storey-type FDR procedure equals to the targeted FDR level. Some consequences of these findings are given in [Section 4](#), which include that a Storey-type FDR procedure is always conservative when the null p-values are independent and uniformly distributed. We end the article with a short discussion in [Section 5](#).

2 Storey-type FDR procedures

Let there be m null hypotheses H_i with associated p-values p_i for $i = 1, \dots, m$, such that only m_0 among them are true nulls and the rest false nulls. However, the true status of each H_i is unknown, and so is the proportion of true nulls $\pi_0 = m_0/m$. A one-step multiple testing procedure (MTP) claims that H_i is a false null if and only if $p_i \leq t$ using a rejection threshold $t \in [0, 1]$. It gives $R(t)$ as the total number of null hypotheses claimed to be false and $V(t)$ as the number of true nulls claimed to be false. The FDR ([Benjamini and Hochberg, 1995](#)) of such an MTP is defined as

$$FDR(t) = \mathbb{E}[V(t) / \max\{R(t), 1\}],$$

where \mathbb{E} is the expectation.

Let $\tilde{\pi}_0(\lambda)$ with a tuning parameter $\lambda \in [0, 1]$ be an estimator of π_0 such that $\tilde{\pi}_0(\lambda)$ ranges in $[0, 1]$. We define a “Storey-type FDR estimator” as

$$\widetilde{FDR}_\lambda(t) = \min \left\{ 1, \frac{\tilde{\pi}_0(\lambda) t}{m^{-1} \max \{R(t), 1\}} \right\} \quad \text{for } t \in [0, 1]. \quad (1)$$

For $\alpha \in [0, 1]$, let

$$t_\alpha(\widetilde{FDR}_\lambda) = \sup \left\{ t \in [0, 1] : \widetilde{FDR}_\lambda(t) \leq \alpha \right\} \quad (2)$$

and the decision rule based on \widetilde{FDR}_λ

$$\text{claim } H_i \text{ as a false null} \iff p_i \leq t_\alpha(\widetilde{FDR}_\lambda). \quad (3)$$

The procedure defined by (1) and (3) is called a “Storey-type FDR procedure”.

A Storey-type FDR procedure allows the use of various estimators of π_0 and is very versatile. For example, (i) it is Storey’s procedure when $\tilde{\pi}_0(\lambda)$ in (1) is the estimator

$$\hat{\pi}_0(\lambda) = \min \left\{ 1, (1 - \lambda)^{-1} m^{-1} \sum_{i=1}^m \mathbf{1}_{\{p_i > \lambda\}} \right\} \quad (4)$$

in [Storey et al. \(2004\)](#); (ii) it is the generalized FDR procedure in [Chen and Doerge \(2014\)](#), designed for multiple testing based on discrete p-values, when $\tilde{\pi}_0(\lambda)$ is the estimator of π_0 proposed there and further studied in [Chen and Doerge \(2015\)](#); (iii) it is the BH procedure in [Benjamini and Hochberg \(1995\)](#) when $\lambda = 0$ is set in (4) in Storey’s procedure or $\tilde{\pi}_0(\lambda) \equiv 1$ is set; (iv) it is a “dynamic adaptive FDR procedure” studied in [Liang and Nettleton \(2012\)](#) when λ in (4) is determined from the p-values and can be a random variable.

3 Stopping time property of the rejection threshold

For notational simplicity, we will from now on write $t_\alpha(\widetilde{FDR}_\lambda)$ as $\tilde{t}_\alpha(\lambda)$. Define the backward filtration

$$\mathcal{F}_t = \sigma(\mathbf{1}_{\{p_i \leq s\}}, t \leq s \leq 1, i = 1, \dots, m) \quad (5)$$

for each $t \in [0, 1)$ and the “stopped backward” filtration $\mathcal{G} = \{\mathcal{F}_{t \wedge \lambda} : 1 \geq t \geq 0\}$. Further, introduce the following

Definition 1. *An estimator $\tilde{\pi}_0(\lambda)$ of the proportion π_0 of true nulls with tuning parameter $\lambda \in [0, 1)$ is said to be regular if it is measurable with respect to (wrt) \mathcal{F}_λ .*

It is crucial to note that, if λ is a functional of the p-values such that the information contained in \mathcal{F}_λ is not sufficient to determine the value of $\tilde{\pi}_0(\lambda)$, then $\tilde{\pi}_0(\lambda)$ is not measurable wrt \mathcal{F}_λ . This can happen for dynamic Storey procedures considered in [Liang and Nettleton \(2012\)](#), where λ is adaptively determined from the data. When this happens, $\tilde{t}_\alpha(\lambda)$ does not have to be a stopping time wrt to \mathcal{G} , the optional stopping theorem can not be applied, and the martingale arguments to prove the conservativeness of (dynamic) Storey FDR procedure in [Storey et al. \(2004\)](#) and [Liang and Nettleton \(2012\)](#) are invalid. Here is our main result:

Theorem 1. *If $\tilde{\pi}_0(\lambda)$ is regular, then $\tilde{t}_\alpha(\lambda)$ is a stopping time with respect to \mathcal{G} . Further, if $\tilde{t}_\alpha(\lambda) < 1$, then*

$$\widetilde{FDR}_\lambda(\tilde{t}_\alpha(\lambda)) = \alpha. \quad (6)$$

The proof of [Theorem 1](#) is given in [Appendix A](#). Note that $\tilde{t}_\alpha(\lambda) < 1$ excludes the meaningless case $\alpha = 1$ as the targeted FDR level. Therefore, $\tilde{t}_\alpha(\lambda) < 1$ in [Theorem 1](#) essentially is not an assumption. [Theorem 1](#) shows that, regardless of whether the p-values are independent, whether they are continuously distributed, or the stochastic orders of their distributions wrt the standard uniform random variable, the rejection threshold of a Storey-type FDR procedure is a stopping time wrt to the stopped backward filtration, when the estimator of the proportion of true nulls is regular. It also shows that a Storey-type FDR estimator equals the pre-specified FDR level at the threshold of its corresponding FDR procedure, when the estimator of the proportion of true nulls is regular and the corresponding decision rule is statistically meaningfully implemented. Equally importantly, [Theorem 1](#) shows that the STP may not hold for the rejection threshold of a Storey-type procedure when $\tilde{\pi}_0(\lambda)$ is not regular, i.e., when adaptivity of the estimator $\tilde{\pi}_0(\lambda)$ to \mathcal{F}_λ is not ensured.

We make four remarks on [Theorem 1](#) and its proof: (i) the measurability of $\tilde{\pi}_0(\lambda)$ wrt to \mathcal{F}_λ is critical to the STP of $\tilde{t}_\alpha(\lambda)$; (ii) the fact that [\(6\)](#), i.e., $\widetilde{FDR}_\lambda(\tilde{t}_\alpha(\lambda)) = \alpha$ for $\tilde{t}_\alpha(\lambda) < 1$,

holds for general p-values is a new result, whose consequences will be discussed in [Section 4](#); (iii) the validity of the STP of $\tilde{t}_\alpha(\lambda)$, regardless of the joint distribution of the p-values, depends crucially on the intrinsic structure of the backward filtration \mathcal{G} ; (iv) the STP of the threshold (wrt a similar backward filtration) defined by equation (7.2) in [Pena et al. \(2011\)](#) does hold trivially due to the special structure of the filtration there.

4 Conservativeness of Storey-type FDR procedures

We give a few implications of [Theorem 1](#). For concise statements, we set assumption A1) as “the null p-values are independent and uniformly distributed”. Further, for each $t \in [0, 1)$ let $\mathcal{H}_t = \sigma(\mathbf{1}_{\{p_i \leq s\}}, 0 \leq s \leq t, i = 1, \dots, m)$.

Corollary 1. *Assume A1), the following hold:*

1. Lemma 4 of [Storey et al. \(2004\)](#) is valid when $\hat{\pi}_0(\lambda)$ in (4) is regular.
2. Storey’s procedure in [Storey et al. \(2004\)](#) is conservative when $\hat{\pi}_0(\lambda)$ in (4) is regular and $\tilde{t}_\alpha(\lambda) < 1$.
3. If λ is a stopping time wrt to $\{\mathcal{H}_t : t \in [0, 1)\}$ and lies in a compact subset $[\kappa, \tau]$ in $(0, 1)$ for some $\kappa < \tau$ and $\tilde{\pi}_0(\lambda)$ is regular, then a Storey-type FDR procedure is conservative when $\tilde{t}_\alpha \leq \kappa$.

Proof. The first claim is a direct consequence of [Theorem 1](#). We show the second claim. By Corollary 1 in [Liang and Nettleton \(2012\)](#), $\mathbb{E}[\widetilde{FDR}_\lambda(\tilde{t}_\alpha)] \geq FDR(\tilde{t}_\alpha)$. However, [Theorem 1](#) implies $\mathbb{E}[\widetilde{FDR}_\lambda(\tilde{t}_\alpha)] = \alpha$. So, $FDR(\tilde{t}_\alpha) \leq \alpha$. Finally, we show the third. By Theorem 4 in [Liang and Nettleton \(2012\)](#), $\mathbb{E}[\widetilde{FDR}_\lambda(\tilde{t}_\alpha)] \geq FDR(\tilde{t}_\alpha)$. Therefore, from [Theorem 1](#) we see $FDR(\tilde{t}_\alpha) \leq \alpha$. This completes the proof. \square

[Corollary 1](#) illustrates the connection between the regularity of the estimator of the proportion of true nulls, the STP of the rejection threshold, and the conservativeness of a Storey-type FDR procedure. The third conclusion in [Corollary 1](#) allows to choose λ in $\tilde{\pi}_0(\lambda)$ from the data in a Storey-type FDR procedure without potentially sacrificing its conservativeness. Further, it

covers the dynamic Storey’s FDR procedure in [Liang and Nettleton \(2012\)](#) where $\hat{\pi}_0(\lambda)$ in (4) is used and λ is determined from the data.

Compared to the proofs of the conservativeness of (dynamic) Storey’s procedure in [Storey et al. \(2004\)](#) and [Liang and Nettleton \(2012\)](#), the proof of [Corollary 1](#) we provide based on [Theorem 1](#) is much shorter and the second conditioning step there is no longer needed. This illustrates the usefulness of our finding (6) for general p-values in the martingale arguments to prove the conservativeness of a Storey-type FDR procedure.

5 Discussion

We have shown the STP of the rejection threshold of a Storey-type FDR procedure for general p-values, that a Storey-type FDR estimator equals the targeted FDR level at its rejection threshold, and that a Storey-type FDR procedure is usually conservative, when the estimator of the proportion of true nulls is regular. This implies that the STP may not hold for the rejection threshold of an adaptive FDR procedure when an estimator of the proportion of true nulls is not adaptive to the backward filtration. Therefore, caution should be taken before applying martingale arguments to prove the conservativeness of an adaptive FDR procedure. In view of the increasing use of stochastic processes (such as decision processes) and suprema related to these processes (such as rejection thresholds) in multiple testing, to better understand the behavior of these suprema remains an important task.

Acknowledgements

This research was funded by a National Science Foundation Plant Genome Research Program grant (No. IOS-1025976) to RWD. Its major part was completed when X. Chen was a PhD student at the Department of Statistics, Purdue University.

A Proofs

In order to prove the STP of the rejection threshold, we first need to understand the the *scaled inverse rejection process* $L(t) = t(\max\{R(t), 1\})^{-1}$ with $t \in [0, 1]$, with which the regularity of $\tilde{\pi}_0(\lambda)$, the property of the stopped backward filtration \mathcal{G} and a contrapositive argument will yield **Theorem 1**. In what follows, no assumption will be made about the independence between the p-values, the continuity of the p-value distributions, or their stochastic orders wrt the standard uniform distribution.

A.1 Downward jumps of the scaled inverse rejection process

Order the p-values into $p_{(1)} < p_{(2)} < \dots < p_{(n)}$ distinctly, where the multiplicity of $p_{(i)}$ is n_i for $i = 1, \dots, n$. Let $p_{(n+1)} = \max\{p_{(n)}, 1\}$ and $p_{(0)} = 0$. Define $T_j = \sum_{l=1}^j n_l$ for $j = 1, \dots, n$.

Lemma A.1. *The process $\{L(t), t \in [0, 1]\}$ is such that*

$$L(t) = \begin{cases} t & \text{if } t \in [0, p_{(1)}), \\ tT_j^{-1} & \text{if } t \in [p_{(j)}, p_{(j+1)}) \text{ for } j = 1, \dots, n-1, \\ tm^{-1} & \text{if } t \in [p_{(n)}, p_{(n+1)}]. \end{cases} \quad (\text{A.1})$$

Moreover, it can only be discontinuous at $p_{(i)}$, $1 \leq i \leq n$, where it can only have a downward jump with size

$$L(p_{(i)}-) - L(p_{(i)}) = \frac{p_{(i)}n_i}{R(p_{(i)})[R(p_{(i)}) - n_i]} > 0.$$

Proof. Clearly

$$R(t) = \begin{cases} 0 & \text{if } 0 \leq t < p_{(1)}, \\ T_j & \text{if } p_{(j)} \leq t < p_{(j+1)}, j = 1, \dots, n-1, \\ m & \text{if } p_{(n)} \leq t \leq p_{(n+1)}, \end{cases}$$

and (A.1) holds. Therefore, the points of discontinuities of $L(\cdot)$ are the original distinct p-values.

This justifies the first part of the assertion.

Now we show that $L(\cdot)$ can only have downward jumps at points of discontinuity. Define

$$\varphi(t, \eta) = \frac{t + \eta}{R(t + \eta)} - \frac{t}{R(t)} = \frac{\eta R(t) + t[R(t) - R(t + \eta)]}{R(t + \eta) R(t)}.$$

From the fact $R(p_{(j)}) - R(p_{(j)}-) = n_j > 0$ but $R(p_{(j)}+) - R(p_{(j)}) = 0$ for each $1 \leq j \leq n$, it follows that $\varphi(p_{(j)}, 0+) = \lim_{\eta \downarrow 0} \varphi(t, \eta) = 0$ but

$$\varphi(p_{(j)}, 0-) = \lim_{\eta \uparrow 0} \varphi(p_{(j)}, \eta) = \frac{p_{(j)} n_j}{R(p_{(j)}) [R(p_{(j)}) - n_j]} > 0.$$

Thus $L(p_{(j)}-) - L(p_{(j)}) = \varphi(p_{(j)}, 0-) > 0$ and the proof is completed. \square

Lemma A.1 shows that the process $L(t)$ is piecewise linear and can only have downward jumps as t increases. The conclusion of **Lemma A.1** is right the contrary to the claim in the proof of Theorem 2 in [Storey et al. \(2004\)](#) that “the process $mt/R(t)$ has only upward jumps and has a final value of 1”, since it says “the process $mt/R(t)$ has only downward jumps”. We construct a counterexample to their claim as follows.

For a small increase c in t which results an increase a_c in $R(t)$, we see that

$$L(t + c) - L(t) = \frac{t + c}{R(t) + a_c} - \frac{t}{R(t)} = \frac{cR(t) - ta_c}{[R(t) + a_c] R(t)} < 0$$

if and only if $\frac{c}{a_c} < \frac{t}{R(t)}$. Construct m p-values with $n \geq 4$ such that there exists some $1 \leq j_0 < n - 2$ with $n_{j_0+1} > T_{j_0}$ but $p_{(j_0+1)} < 1$. Choose c_1 and c_2 such that $0 < c_1 < (p_{(j_0+1)} - p_{(j_0)})/2$ and $0 < c_2 < p_{(j_0+1)} - 2c_1$. Let $t_0 = p_{(j_0+1)} - c_1$ and $c = c_1 + c_2$. Then $p_{(j_0)} < t_0 < p_{(j_0+1)}$, $R(t_0) = T_{j_0}$ and $R(t_0 + c) = T_{j_0} + n_{j_0+1}$. Further, $0 < c < t_0$ and $a_c = n_{j_0+1}$. So $\frac{c}{n_{j_0+1}} < \frac{t_0}{T_{j_0}}$ and $L(t_0 + c) - L(t_0) < 0$. Letting $c \rightarrow 0$ gives $p_{(j_0+1)}$ as a point of downward jump for $L(t)$.

A.2 Proof of **Theorem 1**

When $\alpha = 0$, $\tilde{t}_\alpha(\lambda) = 0$ if $\tilde{\pi}_0(\lambda) > 0$ but $\tilde{t}_\alpha(\lambda) = 1$ if $\tilde{\pi}_0(\lambda) = 0$. In this case, $\tilde{t}_\alpha(\lambda)$ is already a stopping time and $\widetilde{FDR}_\lambda(\tilde{t}_\alpha(\lambda)) = \alpha = 0$. On the other hand, $\tilde{t}_\alpha(\lambda) = 1$ when $\alpha = 1$. In this case, $\tilde{t}_\alpha(\lambda)$ is a stopping time and $\widetilde{FDR}_\lambda(\tilde{t}_\alpha(\lambda)) = \tilde{\pi}_0(\lambda)$. Note that $\tilde{\pi}_0(\lambda) = 0$ can not happen when $0 < \tilde{t}_\alpha(\lambda) < 1$.

First, we show the claim on the stopping time property. By the previous discussion, we only need to consider the case where $0 < \tilde{t}_\alpha(\lambda) < 1$ and $\tilde{\pi}_0(\lambda) > 0$. Define

$$\mathbf{X}_t^{(m)}(\omega) = (\mathbf{1}_{\{p_1 \leq t\}}(\omega), \dots, \mathbf{1}_{\{p_m \leq t\}}(\omega)), \omega \in \Omega,$$

where $(\Omega, \mathcal{A}, \mathbb{P})$ is the probability space with Ω the sample space, \mathcal{A} the sigma-algebra, and \mathbb{P} the probability measure. Then $\mathcal{F}_t = \sigma(\mathbf{X}_s^{(m)}(\omega), 1 \geq s \geq t)$, $t \in [0, 1]$ and $\{\mathcal{F}_t : 0 \leq t \leq 1\}$ is a non-increasing sequence of sub-sigma-algebras of \mathcal{A} . Write $\widetilde{FDR}_\lambda(t) = \widetilde{FDR}_\lambda(t, \omega)$.

By the definition of $\tilde{t}_\alpha(\lambda)$,

$$\{\omega \in \Omega : \tilde{t}_\alpha(\lambda) \leq s\} = \bigcap_{\{t:s < t \leq 1\}} A_t = \tilde{A}_s, \quad (\text{A.2})$$

where $A_t = \{\omega \in \Omega : \widetilde{FDR}_\lambda(t, \omega) > \alpha\}$, $1 \geq t > s$. Thus, we only need to show $\tilde{A}_s \in \mathcal{F}_{s \wedge \lambda}$. Since either $\mathcal{F}_{s \wedge \lambda} = \mathcal{F}_s \supseteq \mathcal{F}_\lambda$ when $s \leq \lambda$ or $\mathcal{F}_{s \wedge \lambda} = \mathcal{F}_\lambda \supseteq \mathcal{F}_s$ when $s \geq \lambda$, the stopping time property holds once we prove $\tilde{A}_s \in \mathcal{F}_s$. Let \mathbb{Q} be the set of all rational numbers. Since $0 < \alpha < 1$ and $\tilde{\pi}_0(\lambda) > 0$, the following decompositions are valid: $A_t = \bigcup_{r \in \mathbb{Q}} (A_{t,r} \cap B_r)$ and

$$\tilde{A}_s = \bigcap_{\{t:s < t \leq 1\}} \bigcup_{r \in \mathbb{Q}} (A_{t,r} \cap B_r) = \bigcup_{r \in \mathbb{Q}} \bigcap_{\{t:s < t \leq 1\}} (A_{t,r} \cap B_r),$$

where $A_{t,r} = \left\{ \omega \in \Omega : \frac{t}{m^{-1} \max\{R(t), 1\}} \geq r \right\}$ and $B_r = \left\{ \omega \in \Omega : \frac{\alpha}{\tilde{\pi}_0(\lambda)} < r \right\}$. Thus, it suffices to show

$$\bigcap_{\{t:s < t \leq 1\}} (A_{t,r} \cap B_r) = \bigcap_{\{t:s < t \leq 1\}} A_{t,r} \cap B_r \in \mathcal{F}_s, \quad (\text{A.3})$$

We now move to show

$$A_{s,r}^* = \bigcap_{\{t:s < t \leq 1\}} A_{t,r} \in \mathcal{F}_s. \quad (\text{A.4})$$

Define $I_i = [p_{(i)}, p_{(i+1)})$, $i = 1, \dots, n-1$. We will add $I_0 = [p_{(0)}, p_{(1)})$ and $I_n = [p_{(n)}, p_{(n+1)}]$ when $p_{(n)} < 1$. When $p_{(n)} = 1$, we take I_{n-1} to be $[p_{(n-1)}, p_{(n)}]$. Obviously there must be a unique j^* with $0 \leq j^* \leq n$ such that $s \in I_{j^*}$. Given $R(1) = m$ and $\tilde{\pi}_0(\lambda) \in [0, 1]$, the properties of $L(\cdot)$

in Lemma A.1 imply

$$A_{s,r}^* = A_{s,r} \cap \left(\bigcap_{j=j^*+1}^{n+1} A_{p(j),r} \right).$$

Consequently, $A_{s,r}^* \in \mathcal{F}_s$, i.e., (A.4) holds.

Recall \tilde{A}_s defined in (A.2). If $B_r \in \mathcal{F}_\lambda$, then (A.4) implies (A.3), which implies $\tilde{A}_s \in \mathcal{F}_s$, i.e., $\tilde{t}_\alpha(\lambda)$ is a STP wrt to \mathcal{G} if $\tilde{\pi}_0(\lambda)$ is measurable wrt to \mathcal{F}_λ . However, this holds since $\tilde{\pi}_0(\lambda)$ is regular.

Finally, we show $\widetilde{FDR}_\lambda(\tilde{t}_\alpha) = \alpha$, where we have written $\tilde{t}_\alpha(\lambda)$ as \tilde{t}_α . Clearly, $\widetilde{FDR}_\lambda(\tilde{t}_\alpha) \leq \alpha$ by the definition of \tilde{t}_α . Our arguments next proceed by contrapositive reasoning, i.e., that $\widetilde{FDR}_\lambda(\tilde{t}_\alpha) < \alpha$ gives a contradiction. Obviously, there must be a unique $0 \leq j' \leq n$ such that $\tilde{t}_\alpha \in I_{j'}$. Since $\tilde{t}_\alpha < 1$, we have $\alpha < 1$, $\tilde{\pi}_0(\lambda) > 0$, $p_{(j'+1)} > 0$ and $I^* = [\tilde{t}_\alpha, p_{(j'+1)}) \subseteq I_{j'}$. Let $\rho_m^*(t) = m^{-1}L(t)$. Then $\rho_m^*(t)$ is continuous and strictly increasing on $I_{j'}$ by Lemma A.1. If $\widetilde{FDR}_\lambda(\tilde{t}_\alpha) < \alpha$ and $\alpha < 1$, then there must be some $d' > 0$ such that $\tilde{I}^* = [\tilde{t}_\alpha, \tilde{t}_\alpha + d'] \subseteq I^*$ and that

$$\widetilde{FDR}_\lambda(t) = \tilde{\pi}_0(\lambda) \rho_m^*(t) \quad \text{for all } t \in \tilde{I}^*. \quad (\text{A.5})$$

However, (A.5) implies that there exists $\hat{t}_\alpha \in \tilde{I}^*$ such that $\hat{t}_\alpha > \tilde{t}_\alpha$ but

$$\widetilde{FDR}_\lambda(\hat{t}_\alpha) = \tilde{\pi}_0(\lambda) \rho_m^*(\hat{t}_\alpha) < \alpha.$$

This contradicts the definition of \tilde{t}_α . Hence $\widetilde{FDR}_\lambda(\tilde{t}_\alpha) = \alpha$ must hold, which completes the whole proof.

References

- Benjamini, Y. and Y. Hochberg (1995). Controlling the false discovery rate: a practical and powerful approach to multiple testing. *J. R. Statist. Soc. Ser. B* 57(1), 289–300.
- Chen, X. and R. Doerge (2014). Generalized estimators for multiple testing: proportion of true nulls and false discovery rate. <http://arxiv.org/abs/1410.4274>.

- Chen, X. and R. Doerge (2015). A weighted fdr procedure under discrete and heterogeneous null distributions. <http://arxiv.org/abs/1502.00973v2>.
- Karatzas, I. and S. E. Shreve (1991). *Brownian Motion and Stochastic Calculus* (2 ed.). Graduate Texts in Mathematics. Springer Verlag.
- Liang, K. and D. Nettleton (2012). Adaptive and dynamic adaptive procedures for false discovery rate control and estimation. *J. R. Statist. Soc. Ser. B* 74(1), 163–182.
- Pena, E. A., J. D. Habiger, and W. Wu (2011). Power-enhanced multiple decision functions controlling family-wise error and false discovery rates. *Ann. Statist.* 39(1), 556–583.
- Storey, J. D., J. E. Taylor, and D. Siegmund (2004). Strong control, conservative point estimation in simultaneous conservative consistency of false discover rates: a unified approach. *J. R. Statist. Soc. Ser. B* 66(1), 187–205.